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A Note on Matrix Riccati Functional Differential Equations

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1. INTRODUCTION

In various problems of the control engineering, mathematical physics, biomathematics, and mathematical biology, we often come across Riccati functional differential equations, either scalar or matrix type. In the scalar case the solution is obtained by the method of quasi-linearization [1]. In this case it has been proved that the successive approximations converge monotonically to the actual solution. Recently Bellman and Vasudevan [2] obtained the iterative approximations to the matrix Riccati differential equations employing the quasi-linearization procedure and have studied the monotonicity and the nature of convergence of successive approximations.

In [2] the solution is obtained by using the Laplace transform treating the coefficient matrices as constant matrices. In this paper we shall be concerned with establishing solutions of the matrix Riccati functional differential equations interms of the fundamental matrices. The approach taken in this paper is substantially new and includes the results of Bellman and Vasedevan [2] as a particular case. The use of Laplace transform becomes difficult when the coefficient matrices are functions of the independent variable t, and hence a new technique has been developed which avoids the use of Laplace transforms. The corollaries obtained in this paper are exactly the results of Bellman and Vasudevan [2].

2. SCALAR RICCATI DIFFERENTIAL EQUATIONS

Let us consider the scalar Riccati differential equation

$$x'(t) = x^2(t) + a(t), \qquad \text{with} \quad x(0) = c. \tag{2.1}$$

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For any admissible function $z(t)$

$$[x(t)-z(t)]^2 \geqslant 0 \Rightarrow x^2(t)=\max_z\,[2x(t)\,z(t)-z^2(t)]$$

or

$$x^2(t) \geqslant 2x(t)\,z(t)-z^2(t).$$

Thus (2.1) can be written as

$$x'(t) \geqslant 2x(t)\,z(t)-z^2(t)+a(t). \tag{2.2}$$

Hence it can be shown that $y(t)$, which is the solution of

$$y'(t)=2y(t)\,z(t)-z^2(t)+a(t) \qquad \text{with} \quad y(0)=c,$$

satisfies the inequality $x(t) \geqslant y(t, z(t))$.

Let us call z as x_0 the initial approximation and y as x_1 the solution at the first approximation. In the second stage we take x_1 as the initial approximation and obtaining x_2, which is the solution of the equation

$$x_2'=2x_2x_1-x_1^2+a(t) \qquad \text{with} \quad x_2(0)=c.$$

Continuing in the same way for $(n+1)$th stage by taking x_n as the initial approximation and obtaining x_{n+1}, which is the solution of the equation

$$x_{n+1}'=2x_{n+1}x_n-x_n^2+a(t), \qquad \text{with} \quad x_{n+1}(0)=c.$$

In this manner we obtain the sequence $x_1, x_2, ..., x_{n+1}$, where

$$x_1 \leqslant x_2 \leqslant x_3 \cdots x_n \leqslant x_{n+1} \leqslant x(t)$$

within the interval of existence of the solution $x(t)$.

This sequence $\{x_n\}$ converges monotonically to x has been shown. It has been proved that the convergence is quadratic, namely,

$$\max_t |x-x_n| \leqslant k_1(\max_t |x-x_{n-1}|^2),$$

for some constant k_1 where the maximum with respect to t is taken in the interval $[0, t]$ [1].

3. Matrix Riccati Equation

In this section, we consider the equation

$$R'(t)=A(t)-R^2(t), \qquad \text{with} \quad R(0)=I, \tag{3.1}$$

where $R(t)$, $A(t)$, I are $n \times n$ matrices whose components are continuous functions on some closed interval J, I being the unit matrix of order n. We also assume that $A(t)$ is positive definite.

In this paper $Y(t)$ stands for a fundamental matrix of $X'(t) = A(t)\, X(t)$ and $Z(t)$ stands for a fundamental matrix of $X'(t) = B^*(t)\, X(t)$.

We first apply the quasi-linearization technique to Eq. (3.1). Consider the identity

$$\begin{aligned} R^2(t) &= [T(t) + R(t) - T(t)]^2 \\ &= T^2(t) + T(t)[R(t) - T(t)] + [R(t) - T(t)]\, T(t) + [R(t) - T(t)]^2. \end{aligned}$$

We therefore conclude that

$$R^2(t) \geqslant T(t)\, R(t) + R(t)\, T(t) - T^2(t)$$

for all symmetric $T(t)$. Note that the equality holds only when $R(t) = T(t)$. Therefore, Eq. (3.1) becomes

$$R'(t) \leqslant A(t) + T^2(t) - T(t)\, R(t) - R(t)\, T(t). \tag{3.2}$$

The associated equation

$$U'(t) = A(t) + T^2(t) - T(t)\, U(t) - U(t)\, T(t), \tag{3.3}$$

with $U_0 = I$.

THEOREM 3.1. *Any solution of*

$$X'(t) = A(t)\, X(t) + X(t)\, B(t) \tag{3.4}$$

is of the form $Y(t)\, C^* Z^*(t)$, *where* C *is a constant square matrix of order* n.

Proof. Let $X(t) = Y(t)\, K(t)$, where $K(t)$ is a square matrix of order n. Then $Y'(t)\, K(t) + Y(t)\, K'(t) = A(t)\, Y(t)\, K(t) + Y(t)\, K(t)\, B(t) \Leftrightarrow K'(t) = K(t)\, B(t) \Leftrightarrow K^{*\prime}(t) = B^*(t)\, K^*(t)$. Since $Z(t)$ is a fundamental matrix of $Z'(t) = B^*(t)\, Z(t)$, it follows that there exists a constant matrix C such that $K^*(t) = Z(t)\, C$ or $X(t) = Y(t)\, C^* Z^*(t)$.

THEOREM 3.2. *Any solution of*

$$X'(t) = A(t)\, X(t) + X(t)\, B(t) + F(t) \tag{3.5}$$

is of the form $X(t) = Y(t)\, C^* Z^*(t) + \bar{Y}(t)$, *where* $\bar{Y}(t)$ *is a particular solution of* (3.5).

Proof. It can easily be verified that $Y(t)\, C^* Z^*(t) + \bar{Y}(t)$ is a solution of (3.5). Now to prove that every solution is of the form, let $X(t)$ be any solution of (3.5) and $\bar{Y}(t)$ be a particular solution of (3.5). Then it can easily be verified that $X(t)-\bar{Y}(t)$ is a solution of (3.4). Hence by Theorem 3.1, we have $X(t)-\bar{Y}(t)=Y(t)\,C^*Z^*(t)$. Therefore, $X(t)=Y(t)\,C^*Z^*(t)+\bar{Y}(t)$.

COROLLARY 3.1. *Let $Y_1(t)$ be a fundamental matrix of $U'(t)=-T(t)\,U(t)$, $Z_1(t)$ be a fundamental matrix of $U'(t)=-T^*(t)\,U(t)$, and $F(t)=A(t)+T^2(t)$. Then any solution of Eq.* (3.3) *is of the form*

$$U(t)=Y_1(t)\,C^*Z_1^*(t)+\bar{Y}(t),$$

where $\bar{Y}(t)$ is a particular solution of (3.3).

Note that this corollary is a generalization of the result proved in [2].

THEOREM 3.3. *A particular solution of $X'(t)=B^*(t)\,X(t)+F^*(t)\,Y^{*^{-1}}(t)$ is of the form*

$$Z(t)\int_a^t Z^{-1}(s)\,F^*(s)\,Y^{*^{-1}}(s)\,ds.$$

Proof. Write $X(t)=Z(t)\,L(t)$. Then one can easily find that $L(t)=\int_a^t Z^{-1}(s)\,F^*(s)\,Y^{*^{-1}}(s)\,ds$.

THEOREM 3.4. *A particular solution $\bar{Y}(t)$ of the non-homogeneous Riccati functional differential equation* (3.5) *is of the form $\bar{Y}(t)=Y(t)[\int_a^t Y^{-1}(s)\,F(s)\,Z^{*^{-1}}(s)\,ds]\,Z^*(t)$.*

Proof. Let $Y(t)$ be a fundamental matrix of $X'(t)=A(t)\,X(t)$, then the matrix $\bar{Y}(t)=Y(t)\,K(t)$ is a solution of (3.5) if and only if $K^{*\prime}(t)=B^*(t)\,K^*(t)+F^*(t)\,Y^{*^{-1}}(t)$. Now by Theorem (3.3) a particular solution of this equation is of the form $K^*(t)=Z(t)\int_a^t Z^{-1}(s)\,F^*(s)\,Y^{*^{-1}}(s)\,ds$. Hence,

$$\bar{Y}(t)=Y(t)\left[\int_a^t Y^{-1}(s)\,F(s)\,Z^{*^{-1}}(s)\,ds\right]Z^*(t).$$

COROLLARY 3.2. *A particular solution of Eq.* (3.3) *is of the form $\bar{Y}(t)=Y_1(t)[\int_a^t Y_1^{-1}(s)[A(s)+T^2(s)]\,Z_1^{*^{-1}}(s)\,ds]\,Z_1^*(t)$.*

THEOREM 3.5. *Any solution of the initial value problem* (3.5) *satisfying the initial condition $X(a)=\alpha$, is given by $X(t)=Y(t)\,Y^{-1}(a)\,\alpha Z^{*^{-1}}(a)\,Z^*(t)+Y(t)[\int_a^t Y^{-1}(s)\,F(s)\,Z^{*^{-1}}(s)\,ds]\,Z^*(t)$, where α is a constant square matrix of order n.*

Proof. By Theorems 3.1., 3.2, and 3.4 the solution of Eq. (3.5) is of the form $X(t)=Y(t)\,C^*Z^*(t)+Y(t)[\int_a^t Y^{-1}(s)\,F(s)\,Z^{*^{-1}}(s)\,ds]\,Z^*(t)$. Substituting the general form of $X(t)$ in the initial condition matrix, we get $Y(a)=Y(a)\,C^*Z^*(a)\Rightarrow C^*=Y^{-1}(a)\,\alpha Z^{*^{-1}}(a)$. Hence,

$$X(t)=Y(t)\,Y^{-1}(a)\,\alpha Z^{*^{-1}}(a)\,Z^*(t)$$
$$+\,Y(t)\left[\int_a^t Y^{-1}(s)\,F(s)\,Z^{*^{-1}}(s)\,ds\right]Z^*(t).$$

COROLLARY 3.3. *Any solution of the initial value problem* (3.3), *satisfying the initial condition* $U(0)=I$ *is given by*

$$U(t)=Y_1(t)\,Y_1^{-1}(0)\,IZ^{*^{-1}}_1(0)\,Z_1^*(t)+Y_1(t)$$
$$\times\left[\int_0^t Y_1^{-1}(s)[A(s)+T^2(s)]\,Z^{*^{-1}}(s)\,ds\right]Z^*(t).$$

4. Monotonicity of the Successive Approximations

Consider Eq. (3.3) with $U(0)=I$. Let $U_1(t)$ be the first approximate estimate of $U(t)$ and take this approximation in the place of $T(t)$, and write the equation for $U_2(t)$, the solution at the second approximation. Then

$$U_2'(t)=A(t)+U_1^2(t)-U_1(t)\,U_2(t)-U_2(t)\,U_1(t),\qquad U_2(0)=I.$$

Continuing in this fashion we construct a sequence of matrix approximations $U_n(t)$, where

$$\begin{aligned}&U'_{n+1}(t)=A(t)+U_n^2(t)-U_n(t)\,U_{n+1}(t)-U_{n+1}(t)\,U_n(t),\\&U_{n+1}(0)=I,\end{aligned}\tag{4.1}$$

Let us now consider the result that the sequence generated is monotone decreasing, that is,

$$U_1(t)\geqslant U_2(t)\geqslant U_3(t)\cdots U_n(t)\geqslant U_{n+1}(t)\geqslant,\,R(t).$$

We have for any n, $R^2(t)\geqslant R(t)\,U_n(t)+U_n(t)\,R(t)-U_n^2(t)$. Hence Eq. (3.1) can be written as

$$R'(t)\leqslant A(t)+U_n^2(t)-R(t)\,U_n(t)-U_n(t)\,R(t).\tag{4.2}$$

Comparing Eq. (4.1) and (4.2) we get $R(t)\leqslant U_{n+1}(t)$, for any n within the interval of existence of the solution $R(t)$; that is, $R(t)$ is the lower bound of the sequence.

THEOREM 4.1. *The solution of the successive approximation of Eq.* (3.3) *form a monotoically decreasing sequence.*

Proof. If we write down the equations for $(n-1)$th, nth approximations to Eq. (3.3), we get

$$U'_n(t) = A(t) + U^2_{n-1}(t) - U_{n-1}(t)\, U_n(t) - U_n(t)\, U_{n-1}(t) \tag{4.3}$$

and Eq. (4.1) with $U_n(0) = U_{n+1}(0) = I$.

Let us now consider the result that the sequence generated is monotone decreasing, $U_1(t) \geqslant U_2(t) \geqslant U_3(t) \cdots U_n(t) \geqslant U_{n+1}(t)$. Now Eq. (4.3) satisfies the inequality

$$U'_n(t) \geqslant A(t) + U^2_n(t) - U_n(t)\, U_n(t) - U_n(t)\, U_n(t). \tag{4.4}$$

From (4.1) and (4.4) we have

$$\begin{aligned}[U_n(t) - U_{n+1}(t)]' \geqslant{} & -U_n(t)[U_n(t) - U_{n+1}(t)] \\ & - [U_n(t) - U_{n+1}(t)]\, U_n(t).\end{aligned} \tag{4.5}$$

Let $Y(t) = U_n(t) - U_{n+1}(t)$. Then the inequality (4.5) can be written as

$$Y'(t) \geqslant -U_n(t)\, Y(t) - Y(t)\, U_n(t), \tag{4.6}$$

with $Y(0) = 0$.

Now the inequality (4.6) can be written as

$$Y'(t) = -U_n(t)\, Y(t) - Y(t)\, U_n(t- + Q(t) \tag{4.7}$$

with $Y(0) = 0$, where $Q(t)$ is positive definite square matrix of order n.

Now using Theorem 3.4, and if $Y_1(t)$ and $Y_2(t)$ are fundamental matrices of $Y'(t) = -U_n(t)\, Y(t)$ and $Y'(t) = -U^*_n(t)\, Y(t)$ then the solution of (4.7) is $Y(t) = Y_1(t)[\int_0^t Y_1^{-1}(s)\, Q(s)\, Y_2^{*^{-1}}(s)\, ds]\, Y_2^*(t)$. Therefore, $U_{n+1}(t) \leqslant U_n(t)$, for all n. Thus the successive approximations form a monotonically decreasing sequence.

The Nature of Convergence

Consider Eq. (4.3) with $U_n(0) = I$, and rewrite it as

$$\begin{aligned}U'_n(t) = {} & A(t) - U^2_{n-1}(t) - U_{n-1}(t) \\ & \times [U_n(t) - U_{n-1}(t)] - [U_n(t) - U_{n-1}(t)]\, U_{n-1}(t)\end{aligned} \tag{4.8}$$

with $U_n(0) = I$. Consider the identity

$$\begin{aligned} R^2(t) &= [U_{n-1}(t) + R(t) - U_{n-1}(t)]^2 \\ &= U_{n-1}^2(t) + U_{n-1}(t)[R(t) - U_{n-1}(t)] \\ &\quad + [R(t) - U_{n-1}(t)]\, U_{n-1}(t) + [R(t) - U_{n-1}(t)]^2. \end{aligned}$$

Let us rewrite Eq. (3.1) using the above identity:

$$\begin{aligned} R'(t) &= A(t) - U_{n-1}^2(t) - U_{n-1}(t)[R(t) - U_{n-1}(t)] \\ &\quad - [R(t) - U_{n-1}(t)]\, U_{n-1}(t) - [R(t) - U_{n-1}(t)]^2, \end{aligned} \tag{4.9}$$

with $R(0) = I$. From (4.8) and (4.9) we have

$$\begin{aligned} [U_n(t) - R(t)]' &= -U_{n-1}(t)[U_n(t) - R(t)] - [U_n(t) - R(t)] \\ &\quad \times U_{n-1}(t) + [R(t) - U_{n-1}(t)]^2, \end{aligned} \tag{4.10}$$

with $(U_n - R)(0) = 0$. Then by using Theorem 3.4 we get the solution of Eq. (4.10) as

$$U_n(t) - R(t) = Z_1(t) \left[\int_0^t Z_1^{-1}(s)[R(s) - U_{n-1}(s)]^2\, Z^{*-1}_2(s)\, ds \right] Z_2^*(t),$$

where $Z_1(t)$ and $Z_2(t)$ are fundamental matrices of

$$[U_n(t) - R(t)]' = -U_{n-1}(t)[U_n(t) - R(t)]$$

and

$$[U_n(t) - R(t)]' = -U_{n-1}^*(t)[U_n(t) - R(t)].$$

In Eq. (4.10) the matrix $Q(t)$ is positive definite and it contains $[R(t) - U_{n-1}(t)]^2$ as a factor.

References

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